NOTES OF SEM-II
CORE-03

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Theorem:— The set N of natural Numbers is not bounded above.

Proof:— Suppose on the Contrary that N is bounded above.

Hence N has a Supremum Say $u = \sup N$
By the theorem of Supremum and Infimum there exists Some $n \in N$ Such that $u-1 < n$.
Then for this n, we have $n+1 > u$
Since $n+1 \in N$ This Contradicts the fact that $u = \sup N$
Hence N is not bounded above.

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Archimedean Property of rational numbers:

Theorem:— If a and b be two positive rational numbers Such that $a > b$, $a \neq 0$ then There exists a positive integer n Such that $nb > a$

Proof: We have to Prove here that there is Some $n \in N$ Such that $nb > a$
Here we take N be the set of natural numbers we suppose on the Countrary. This means that we have to Prove now that $nb < a$ for all $n \in N$
Therfore we suppose that $nb < a$ for all $n \in N$

Then $nb < a$

$\Rightarrow \frac{nb}{a} < 1$

$\Rightarrow \frac{nb}{a} < m$ for all $m \in N$

$\Rightarrow \frac{nb}{m} < a$

$\Rightarrow \frac{n}{m} < a/b$ — (1)

But $\frac{n}{m}$ is a positive rational number, say x

Hence (1) $\Rightarrow x < a/b$ for all $x \in Q^+$

where Q^+ denotes the set of positive rational numbers and this is false.

Thus on the supposition that $nb < a$ for all $n \in N$ we arrive at a contradiction.

Thus we prove the theorem.

Density Theorem

Statement:- There are infinity of rational numbers between any two different rational number.

Proof:- Let a and b be any two different numbers and let $a < b$ — (1)

Suppose that K is any positive rational number.

Then since $K > 0$ we have $aK < bK$

$\Rightarrow a + aK < a + bK$

$\Rightarrow a(1+K) < a + bK$ — (2)

$\therefore a < \frac{a+bK}{1+K}$ — (3)

Again adding bK to both sides of (1), we get

$a + bK < b + bK$

$\Rightarrow a + bK < b(1+K)$

$\therefore \frac{a+bK}{1+K} < b$ — (4)

Hence from (3) and (4), we get—

$$a < \frac{a+bK}{1+K} < b \quad \text{— (5)}$$

Now, $\frac{a+bK}{1+K}$ is a rational number which lies between ~~a~~ a and b and this is true for every positive rational number K.

But K is our choice and we can assign infinity of values to K. Hence corresponding to infinite values of K, we find that $\frac{a+bK}{1+K}$ can pick up infinity of values.

Each value is a rational number which lies between a and b

Hence we prove the theorem.

Cantor's theorem on the nest of Intervals:-

Statement: If $\{I_n = [a_n, b_n]\}$, $n = 1, 2, 3 \cdots$ denotes a sequence of closed intervals,

(I) each of which is contained in the preceding interval i.e. $I_{n+1} \subseteq I_n$ and

(II) If the lengths of the intervals of the sequence converge to zero

i.e. $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$ then there exists one and only one point ξ which lies inside every interval of the sequence of the intervals such that ~~$Lt\, a_n = Lt$~~

$$Lt\, a_n = Lt\, b_n = \xi$$

i.e. $\bigcap_{n=1}^{\infty} I_n$ is a singleton.

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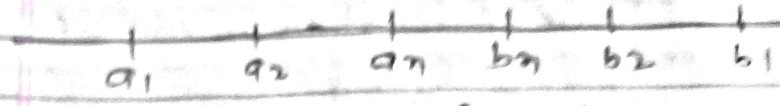
Proof: By hypothesis $I_n = [a_n, b_n]$

Then since $I_n \supseteq I_{n+1}$, we have

$a_n \leq a_{n+1}$, $b_n \geq b_{n+1}$ $(n = 1, 2, 3, \ldots)$

Hence $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$ (1)

$b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots$ (2)



The sequence (1) viz $\{a_n\}$ is monotonic increasing and is bounded above by b_1.

Hence the sequence $\{a_n\}$ is convergent.

Let the limit of $\{a_n\}$ be α.

Similarly the sequence (2) viz $\{b_n\}$ is monotonic decreasing and is bounded below by a_1.

We assume that a monotonic increasing sequence which is bounded above tends to a limit and that a monotonic decreasing sequence which is bounded below tends to a limit.

Hence the sequence $\{b_n\}$ is convergent.

Let the limit of $\{b_n\}$ be β.

Thus we have $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$

Next, we want to show that $\alpha = \beta$.

We have $b_n = b_n - a_n + a_n$ for all n

$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} (b_n - a_n) + \lim_{n\to\infty} a_n$

$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$, since $\lim_{n\to\infty} (b_n - a_n) = 0$

$\therefore \beta = \alpha = \xi$ (say)

$\therefore$ the sequence $\{a_n\}$ and $\{b_n\}$ converge to the same limit ξ

Now, since ξ is the limit of the sequence $\{a_n\}$ which is monotonic increasing and is bounded above, therefore ξ is the ~~least upper bound~~ of $\{a_n\}$

Similarly ξ is the g.l.b of $\{b_n\}$

Thus we have $a_n \leq \xi$ and $b_n \geq \xi$

$\therefore a_n \leq \xi \leq b_n$ for all n

which shows that the point ξ lies inside every interval of the sequence $\{I_n\}$

i.e. $a_n \leq \xi \leq b_n$ for all n

i.e. $\xi \in I_n$ for all n

i.e. $\xi \in \bigcap_{n=1}^{\infty} I_n$

We shall now prove that ξ is unique. For, if not let ξ' be another such number, then we have $a_n \leq \xi' \leq b_n$ for all n

$\therefore \lim_{n\to\infty} a_n \leq \xi' \leq \lim_{n\to\infty} b_n$

or $\xi \leq \xi' \leq \xi$

Thus we must have $\xi = \xi'$ which shows that ξ is unique.

This proves the theorem

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State and Prove Bolzano-Weierstrass's Theorem

Statement:— Bounded infinite set has at least one limit point.

Proof:— Let S be a given bounded infinite set, Since S is bounded, there exists a closed Interval $I = [a, b]$ such that $S \subset]a, b]$

Now we bisect the interval $[a, b]$ by the point $c = \frac{a+b}{2}$

Now the closed sub-interval $[a, c]$ and $[c, b]$ are such that at least one of them must contain an infinite number of points of S, for other wise if both the interval contain finite number of points of S, then S would be a finite set.

Thus we have two alternatives viz. one of these sub-intervals contains an infinity of points of S or both of them contain an infinity of points of S. In any case we select a sub interval which contains an infinity of points of S and let us denote it by $I_1 = [a_1, b_1]$

Next we bisect the interval $[a_1, b_1]$ by the point $c_1 = \frac{a_1+b_1}{2}$

Again of the two sub-interval $[a_1, c_1]$ and $[c_1, b_1]$ we select a sub-interval which contains an infinity of elements of S and let us denote it by $I_2 = [a_2, b_2]$

Continuing this process indefinitely we get a sequence of nested intervals, viz. $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$ i.e $I \supset I_1 \supset I_2 \cdots$ such that each of these interval contains an infinity of points of S.

Also $b_n - a_n = \frac{b-a}{2^n}$

$\therefore$ The length of the Interval $[a_n, b_n] \to 0$ as $n \to \infty$

Consequently by Cantor's theorem there exists one and only one point α which is common to every Interval of the Sequence of Intervals $\{[a_n, b_n]\}$ such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha$$

Finally we shall prove that α is a limit point of the set S.

For this, we take an arbitrary open Interval (h, k) which contains α.

We can now find an Integer m such that $[a_m, b_m] \subset (h, k)$.

Now since $[a_m, b_m] \in \{[a_n, b_n]\}$ it follows that $[a_m, b_m]$ contains an infinity of points of S and consequently the arbitrary open Interval (h, k) which contains α also contains an infinity of points of S.

Thus α is limit points of the set S.

Sequences and their limits

Theorem:— Every Convergent Sequence is bounded.

Proof:— Let $\{a_n\}$ be a Convergent Sequence and let $a_n \to l$ as $n \to \infty$

That is $|a_n - l| < \varepsilon$ when $n \geq m$

Thus we have $l - \varepsilon < a_n < l + \varepsilon$ when $n \geq m$

Let s_p and s_q be the least and greatest of the numbers $a_1, a_2, a_3 \ldots a_n$

Then if K be the smaller of the two numbers s_p, $l - \varepsilon$ and K the greater of the two numbers s_q, $l + \varepsilon$ then it is clear that $-K < a_n < K$ for all values of n.

Thus the sequence $\{a_n\}$ is bounded.

If λ is the greater of the two numbers $|K|$ and $|K|$, we have $|a_n| < \lambda$ for all values of n.

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Theorem:— State and Prove Cauchy's General Principle of Convergence.

Statement: The necessary and sufficient Condition for the Convergence of any sequence $\{u_n\}$ is the Corresponding to every arbitrary ϵ there exists a positive integer m such that

$$|u_{n+p} - u_n| < \epsilon$$

for all values of $n \geq m$ and for all positive Integral values of p.

Proof:— The Condition is necessary.

Let $u_n \to l$ as $n \to \infty$

DATE: / /
PAGE:

Then $|u_n - l| < \frac{\epsilon}{2}$ when $n \geq m$

Since $n+p > n$ therefore $|u_{n+p} - l| < \frac{\epsilon}{2}$
where p is any positive Integer.

$\therefore |u_{n+p} - u_n| = |(u_{n+p} - l) - (u_n - l)|$
$\leq |u_{n+p} - l| + |u_n - l|$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
where $n \geq m$ and p is any positive integer

The Condition is Sufficient

Let $|u_{n+p} - u_n| < \epsilon$ when $n \geq m$ and p is any +ve integer.

Then $u_n - \epsilon < u_{n+p} < u_n + \epsilon$ when $n \geq m$ and p is any +ve integer.

Thus $\{u_{n+p}\}$ is bounded when $p \to \infty$

If N and M be the lower and upper bounds respectively, then $N \geq u_n - \epsilon$ and $M \leq u_n + \epsilon$

$\therefore M - N \leq u_n + \epsilon - u_n + \epsilon = 2\epsilon$

Since ϵ is an arbitrarily small positive number
Therefore $M - N = 0$ i.e $M = N$

$\therefore M - \epsilon < u_{n+p} < M + \epsilon$

i.e $|u_{n+p} - M| < \epsilon$

i.e $u_{n+p} \to M$ as $p \to \infty$

Hence $\{u_n\}$ Converges to M.

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Q. Discuss the Convergence of the Sequence $\{a_n\}$ defined by $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

Here $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n-2} + \frac{1}{n+n-1} + \frac{1}{n+n}$ (n terms)

$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{(n+1)+(n+1)-2}$

$+ \frac{1}{n+1+(n+1)-1} + \frac{1}{n+1+n+1}$, ($n$ terms)

$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$= \frac{\cancel{2n} + \cancel{2} + \cancel{2n} + 1 - \cancel{4n} - \cancel{2}}{(2n+1)(2n+2)}$

$= \frac{1}{(2n+1)(2n+2)} > 0$ for all n

or $a_{n+1} > a_n$ for all n.

Hence the Sequence $\{a_n\}$ is monotone increasing

Again $a_n < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

$a_n < n \times \frac{1}{n} = 1$ for all n.

Hence the Sequence $\{a_n\}$ is bounded above and monotone increasing and therefore it must Converge.

DATE: / /
PAGE:

). Prove that the sequence $\{a_n\}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \text{ diverges}$$

Here $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$

Changing n to $n+1$ we get

$$a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$$

$\therefore a_{n+1} - a_n = \frac{1}{n+1} > 0$ for all n.

or $a_{n+1} > a_n$.

Therefore the sequence $\{a_n\}$ is Monotonic increasing.

Now $a_{n+p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+p}$; $p \in N$

$\therefore |a_{n+p} - a_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} \right|$

Taking $p = n$ we get—

$$|a_{2n} - a_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right|$$

or $|a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, as $|x| = x$ when $x > 0$

or $|a_{2n} - a_n| > \frac{1}{n+n} + \frac{1}{n+n} + \cdots + \frac{1}{2n}$

or $|a_{2n} - a_n| > \frac{n}{2n} = 1/2$ for all n —— ①

Let us suppose that the sequence $\{a_n\}$ is Convergent, Therefore, by Cauchy's General principle of Convergence, given $\epsilon > 0$, however small, there exists a positive integer m, such that

$$|a_{n+p} - a_n| < \epsilon \text{ for } n \geq m \text{ and } p \in N$$

Taking $p = m = n$, we get $|a_{2n} - a_n| < \epsilon$ for all n —— ②

Let ϵ be such that $0 < \epsilon < 1/2$

Then from ①

$$|a_{2n} - a_n| > 1/2 > \epsilon$$

This is Contradiction to ②

Therefore our supposition is wrong.

Hence $\{a_n\}$ Cannot Converge.

We find that the sequence $\{a_n\}$ is monotone increasing and does not Converge and therefore it must diverge

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Convergency and Divergency of Series

Theorem: — First Form of Comparison Test.

Statement: — If after a particular term, the ratio of the corresponding terms of two series of positive terms be always finite and non-zero, the series will be both convergent or both divergent.

Proof: — Let Σu_n and Σv_n be the two given series whose $\frac{u_n}{v_n}$ is always finite for $n \geq m$, where m is a particular integer.

Let $\frac{u_r}{v_r}$ be the greatest fraction and $\frac{u_t}{v_t}$ the least fraction.

Also let $\frac{u_r}{v_r} = K_1$, $\frac{u_t}{v_t} = K_2$.

$\therefore$ $\frac{u_m}{v_m}, \frac{u_{m+1}}{v_{m+1}}$ ..... are all less than or equal to K_1.

$\therefore$ $u_n \leq K_1 v_n$ for $n \geq m$

i.e. $u_m \leq K_1 v_m$

$u_{m+1} \leq K_1 v_{m+1}$

$u_{m+2} \leq K_1 v_{m+2}$

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$u_n \leq K_1 v_n$

Adding we get

$$u_m + u_{m+1} + u_{m+2} + \cdots + u_n \leq K_1 (v_m + v_{m+1} + v_{m+2} + \cdots + v_n)$$

i.e. $$\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} \leq K_1$$

Similarly $$\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} \geq K_2$$

But K_1 and K_2 are finite

So let $\frac{u_m + u_{m+1} + \cdots + u_n}{v_m + v_{m+1} + \cdots + v_n} = L$, where L is finite which lies between K_1 and K_2

$\therefore$ $u_m + u_{m+1} + \cdots + u_n = L(v_m + v_{m+1} + \cdots + v_n)$ for all n

This shows that $u_m + u_{m+1} + \cdots$ and $v_m + v_{m+1} + \cdots$

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Converge or diverge together.

So adding a finite number of terms we find that $u_1+u_2+\cdots$ and $v_1+v_2+\cdots$ Converge or diverge together.

Theorem:— Prove that the infinite series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots\infty$ is Convergent if $p>1$ and divergent if $p\leq 1$.

Proof:. Let the given series is ~~bounded~~ denoted by S.

Since S is a series of positive terms its Convergence or divergence is not affected by grouping the terms in brackets in any way we please.

Case I Let $p>1$

Group the terms as follows:

$$1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\cdots \text{ to } \infty$$

evidently $\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}$ i.e, $\frac{2}{2^p}$

$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}$ i.e $\frac{4}{4^p}$

Adding, we obtain $S<1+\frac{2}{2^p}+\frac{4}{4^p}+\cdots$ to ∞

which is in g.p. where Common ratio is $\frac{2}{2^p}<1$ for $p>1$

i.e $S<\frac{1}{1-2/2^p}$ Hence S is Convergent.

Case II Let $p=1$

then the given series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ to ∞

Group the terms as follows.

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \text{ to } \infty$$

obviously $1+\frac{1}{2}=1+\frac{1}{2}$

$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}$ i.e. $\frac{2}{4}$ i.e $\frac{1}{2}$

Adding, we get $S>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ to ∞

whereas divergent series

Hence S is divergent.

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Q. Determine the convergency or divergency of the following series.

$$1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+\cdots \text{ to } \infty \text{ for all values of } p$$

Sol:- Let the nth term of the given series be denoted by u_n. Then $u_n = \frac{n^p}{\lfloor n}$

Replacing n by $n+1$, we get $u_{n+1} = \frac{(n+1)^p}{\lfloor n+1}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)\lfloor n} \times \frac{\lfloor n}{n^p} = \left(1+\frac{1}{n}\right)^p \times \frac{1}{1+n}$$

$$\text{or } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

for all values of p

Hence, by D'Alembert's ratio test, the given series is convergent for all value of p

Q. Discuss the convergence of the series

$$1+\frac{2^{-p}}{\lfloor 2}+\frac{3^{-p}}{\lfloor 3}+\cdots+\frac{n^{-p}}{\lfloor n}+\cdots \text{ to } \infty$$

Sol:- Let the nth term of the given series be denoted by u_n. Then $u_n = \frac{n^{-p}}{\lfloor n}$

$$\therefore u_{n+1} = \frac{(n+1)^{-p}}{\lfloor n+1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^{-p}}{\lfloor n+1} \times \frac{\lfloor n}{n^{-p}} = \frac{(1+1/n)^{-p}}{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+1/n)^{-p}}{n+1} = 0 < 1$$

Hence, by D'Alemberts ratio test, the given series is convergent.

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Theorem: State and Prove Cauchy's Root (nth root) Test for Convergence of infinite Series.

Statement:- A series $\sum u_n$ of positive terms is Convergent if $u_n^{1/n} < K < 1$ where K is a fixed number or n has any value. And the series is divergent if $u_n^{1/n} \geq 1$ for all n.

Proof:- When $u_n^{1/n} < K < 1$ for all n, the $u_n < K^n$

$\therefore u_1 + u_2 + u_3 + \cdots < K + K^2 + K^3 + \cdots$

As $K < 1$, the series on the right is Convergent and hence the series $u_1 + u_2 + u_3 + \cdots$ is Convergent.

When $u_n^{1/n} \geq 1$ then for all n, $u_n \geq 1$

$\therefore u_1 + u_2 + \cdots + u_n \geq 1 + 1 + 1 + \cdots$ to n terms $= n$

$\therefore u_1 + u_2 + u_3 + \cdots \geq n$ when n tends to ∞, the series $\sum u_n$ is divergent.

Theorem: — Leibnitz's Test:

Prove that the alternating series

$$\sum_{1}^{\infty} (-1)^{n-1} u_n$$

i.e. $u_1 - u_2 + u_3 - u_4 + \cdots + \infty$

is convergent if

(i) each term is numerically less than the preceding i.e. $|u_{n+1}| < |u_n|$, for all n, and

(ii) $u_n \to 0$ as $n \to \infty$

Proof:- Let $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n})$ — (A)

and this can be written in the form

$S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n}$ — (B)

Since $u_1 > u_2 > u_3 \cdots\cdots$ therefore.

$u_1 - u_2,\ u_3 - u_4, \cdots$ all are positive.

Also $u_2 - u_3,\ u_4 - u_5 \cdots\cdots$ all are positive.

From (A) it, therefore, follows that S_{2n} is positive and increases with n, and from (B) it is seen that S_{2n} is always less than the fixed number u_1, Hence S_{2n} tends to a limit which is less than u_1.

Again $S_{2n+1} = S_{2n} + u_{2n+1}$

$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$

Since $\lim_{n\to\infty} u_{2n+1} = 0$

$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$

The series is therefore convergent its sum is positive and less than u_1.

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DATE: / /
PAGE:

Theorem: Comparison Test.

(I) If $\sum u_n$ and $\sum v_n$ be two positive term series and after some particular term $\frac{u_n}{u_{n-1}} < \frac{v_n}{v_{n-1}}$ then $\sum u_n$ converges provided that $\sum v_n$ converges.

Proof:- Let the series beginning after some particular term be

$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ to ∞

and $v_1 + v_2 + v_3 + \cdots + v_n + \cdots$ to ∞

It is given that
$$\frac{u_2}{u_1} < \frac{v_2}{v_1}$$
$$\frac{u_3}{u_2} < \frac{v_3}{v_2}$$
$$\cdots\cdots$$
$$\frac{u_n}{u_{n-1}} < \frac{v_n}{v_{n-1}}$$

Multiplying all these inequalities, we get—
$$\frac{u_2}{u_1} \times \frac{u_3}{u_2} \cdots \frac{u_n}{u_{n-1}} < \frac{v_2}{v_1} \times \frac{v_3}{v_2} \cdots \frac{v_n}{v_{n-1}}$$

or $\frac{u_n}{u_1} < \frac{v_n}{v_1}$ or $u_n < \left(\frac{u_1}{v_1}\right) v_n$

Let $\frac{u_1}{v_1} = K$, then $u_n < K v_n$

or $\sum u_n < K \sum v_n$.

If $\sum v_n$ converges, then $\sum u_n$ also converges.

(II) If $\sum u_n$ and $\sum v_n$ be two positive terms series and after some particular term $\frac{u_n}{u_{n-1}} > \frac{v_n}{v_{n-1}}$ then $\sum u_n$ diverges provided that $\sum v_n$ diverges.

Proof:- Let the series beginning after some particular term be

$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ to ∞

and $v_1 + v_2 + v_3 + \cdots + v_n + \cdots$ to ∞

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It is given that $\frac{u_2}{u_1} > \frac{v_2}{v_1}$

$$\frac{u_3}{u_2} > \frac{v_3}{v_2}$$

$$\cdots\cdots$$

$$\frac{u_m}{u_{m-1}} > \frac{v_m}{v_{m-1}}$$

Multiplying all these inequalities, we get

$$\frac{u_2}{u_1} \times \frac{u_3}{u_2} \cdots \frac{u_m}{u_{m-1}} > \frac{v_2}{v_1} \times \frac{v_3}{v_2} \cdots \frac{v_m}{v_{m-1}}$$

or $\frac{u_m}{u_1} > \frac{v_m}{v_1}$ or $u_m > \left(\frac{u_1}{v_1}\right) v_m$

Let $\frac{u_1}{v_1} = K$

Then $u_m > K \cdot v_m$

or $\Sigma u_m > K \Sigma v_m$

If Σv_m is divergent, then Σu_m is also divergent.

DATE: / /
PAGE:

Theorem:— State and Prove Raabe's Test.

Statement:— The positive term series $\sum u_n$ converges or diverges according as $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} > \text{ or } < 1$

Proof:— Let us compare the given series $\sum u_n$ with the auxillary series $\sum v_n$.

where $\sum v_n = \frac{1}{\sum n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\frac{1}{(n+1)^p}+\cdots$

We know that $\sum v_n$ is convergent or divergent according as $p >$ or < 1

Hence, by comparision test, the given series $\sum u_n$ will be convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

or $$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{1/n^p}{\frac{1}{(n+1)^p}}$$

or $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p$$

or $$\frac{u_n}{u_{n+1}} > \text{ or } < 1+p\times\frac{1}{n}+\frac{p(p-1)}{\lfloor 2}\times\frac{1}{n^2}+\cdots$$ (By Binomial th)

or $$\frac{u_n}{u_{n+1}}-1 > \text{ or } < \frac{1}{n}\left[p+\frac{p(p-1)}{\lfloor 2}\times\frac{1}{n}+\cdots\right]$$

or $$n\left(\frac{u_n}{u_{n+1}}-1\right) > \text{ or } < p+\frac{p(p-1)}{\lfloor 2}\times\frac{1}{n}+\cdots$$

Proceeding to limits as $n\to\infty$, we get

$$\lim_{n\to\infty}\left\{n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right\} > \text{ or } < p$$

or $$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} > \text{ or } < 1$$

Hence the theorem

DATE: / /
PAGE:

Theorem:- Logarithmic Test

Statement:- The positive term series Σu_n converges or diverges according as $\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > \text{ or } < 1$

Proof:- Let us compare the given series Σu_n with the auxilliary series Σv_n,

where $\Sigma v_n = \Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \frac{1}{(n+1)^p}$ to ∞

we know that Σv_n is convergent or divergent according as $p >$ or < 1

Hence, by comparison test, the given series Σu_n will be convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}}$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p$$

$$\text{or } \log\frac{u_n}{u_{n+1}} > \text{ or } < p\log\left(1+\frac{1}{n}\right)$$

$$\text{or } \log\frac{u_n}{u_{n+1}} > \text{ or } < p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \text{ to } \infty\right)$$

$$\text{or } n\log\frac{u_n}{u_{n+1}} > \text{ or } < p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots \text{ to } \infty\right)$$

Proceeding to limits as $n\to\infty$, we get—

$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > \text{ or } < p$$

$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > \text{ or } < 1$$

Hence the test

~~Hence the Result~~

Theorem:- State and Prove Cauchy's Condensation Test.

Statement: If the function $f(n)$ is positive for all integral values of n, and continually decreases as n increases and if a is a positive integer greater than unity, then the two series $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together.

Proof:- The given series are

$$f(1)+f(2)+f(3)+\cdots+f(n)+\cdots \text{ to } \infty \quad (1)$$

and $$a f(a)+a^2 f(a^2)+a^3 f(a^3)+\cdots+a^n f(a^n)+\cdots \text{ to } \infty \quad (2)$$

Let S_1 and S_2 denote the sums of the series (1) and (2) respectively. Let us arrange the terms of the series (1) as follows:-

$$\{f(1)+f(2)+\cdots+f(a)\}+\{f(a+1)+f(a+2)+\cdots+f(a^2)\}$$
$$+\{f(a^2+1)+f(a^2+2)+\cdots+f(a^3)\}$$
$$+\cdots\cdots$$
$$+\{f(a^{r-1}+1)+f(a^{r-1}+2)+\cdots+f(a^r)\}+\cdots\cdots$$

The rth group of this series is $f(a^{r-1}+1)+f(a^{r-1}+2)+\cdots+f(a^r)$

This group contains (a^r-a^{r-1}) terms, that is $a^{r-1}(a-1)$.

Since the function $f(n)$ continually decreases as n increases, therefore

$$f(a^{r-1}+1) < f(a^{r-1})$$
$$f(a^{r-1}+2) < f(a^{r-1})$$
$$\cdots\cdots$$
$$f(a^r) < f(a^{r-1})$$

Adding $f(a^{r-1}+1)+f(a^{r-1}+2)+\cdots+f(a^r) < a^{r-1}(a-1)f(a^{r-1})$.

Putting $r=1,2,3\ldots$ successively, we get

$$f(2)+f(3)+\cdots+f(a) < (a-1)f(1)$$
$$f(a+1)+f(a+2)+\cdots+f(a^2) < a(a-1)f(a)$$
$$\cdots\cdots$$

Adding $S_1-f(1) < (a-1)[f(1)+S_2]$

or $S_1 < af(1)+(a-1)S_2$

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This shows that S_1, that is, $\sum f(n)$ is Convergent, Provided S_2, that is $\sum a^n f(a^n)$ is Convergent, as $a-1$ and $f(1)$ are finite.

Again $f(a^{r-1}+1) > f(a^r)$

$f(a^{r-1}+2) > f(a^r)$

$\cdots\cdots\cdots$

$f(a^r) = f(a^r)$

Adding $f(a^{r-1}+1)+f(a^{r-1}+2)+\cdots+f(a^r) > \frac{a-1}{a} a^r f(a^r)$

Putting $r=1,2,3\ldots$ successively, we get

$f(2)+f(3)+\cdots+f(a) > \frac{a-1}{a} a f(a)$

$f(a+1)+f(a+2)+\cdots+f(a^2) > \frac{a-1}{a} a^2 f(a^2)$

$\cdots\cdots\cdots$

Adding $S_1 - f(1) > \frac{a-1}{a} S_2$

If S_2, that is $\sum a^n f(a^n)$ is divergent, then S_1, that is $\sum f(n)$ is divergent as $a-1$ and $f(1)$ are finite.

DATE: / /
PAGE:

Theorem:— State and prove Kummer's test.

Statement: If $\sum u_n$ and $\sum \frac{1}{d_n}$ be two series of positive terms, where $\sum \frac{1}{d_n}$ is divergent and if $v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}$ then

(I) The series $\sum u_n$ is convergent provided there exists a number p such that $v_n \geq K > 0$, for all $n \geq p$ and K is a fixed number.

(II) The series $\sum u_n$ is divergent provided there exists a number q such that $v_n \leq 1 < 0$ for all $n \geq q$.

Proof:- (I) Here $v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}$

We have $v_n \geq K > 0$ for all $n \geq p$

or $d_n \frac{u_n}{u_{n+1}} - d_{n+1} \geq K$ for all $n \geq p$

or $d_n u_n - d_{n+1} \cdot u_{n+1} \geq K u_{n+1}$, for all $n \geq p$

Putting $n = p, p+1 \ldots (n-1)$ successively

we get $d_p u_p - d_{p+1} u_{p+1} \geq K u_{p+1}$

$d_{p+1} u_{p+1} - d_{p+2} \cdot u_{p+2} \geq K u_{p+2}$

$d_{n-1} u_{n-1} - d_n u_n \geq K u_n$

Adding we get $d_p u_p - d_n u_n \geq K (u_{p+1} + u_{p+2} + \cdots + u_n)$

or $u_{p+1} + u_{p+2} \cdots + u_n \leq \frac{1}{K}(d_p u_p - d_n u_n) < \frac{1}{K} d_p u_p$

as $u_n d_n > 0$ as $K > 0$

or $\{u_1 + u_2 + \cdots + u_p + u_{p+1} + u_{p+2} + \cdots + u_n\} - \{u_1 + u_2 + \cdots + u_p\} < \frac{1}{K} d_p u_p$

Let the sum of first r terms of $\sum u_n$ be denoted by S_r.

then $S_n - S_p < \frac{1}{K} d_p u_p$

or $S_n < \frac{1}{K} d_p u_p + S_p$.

Proceeding to limits as $n \to \infty$, we get

$\lim_{n \to \infty} S_n < \frac{1}{K} d_p u_p + S_p$

or $\sum u_n = \frac{1}{K} d_p u_p + S_p$

But $\frac{1}{K} d_p u_p + S_p$ is a fixed number.

Hence $\sum u_n$ is convergent

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(ii) We have $v_n \le l < 0$ for all $n \ge q$

or $d_n \frac{u_n}{u_{n+1}} - d_{n+1} < 0$ for all $n \ge q$

or $d_n u_n < d_{n+1} u_{n+1}$ for all $n \ge q$.

Putting $n = q, q+1, q+2, \ldots, n-1$ successively, we get

$d_q u_q < d_{q+1} u_{q+1}$, $d_{q+1} u_{q+1} < d_{q+2} u_{q+2} \ldots$

$d_{n-1} u_{n-1} < d_n u_n$.

$\therefore d_q u_q < d_{q+1} u_{q+1} < d_{q+2} u_{q+2} < \ldots < d_n u_n$

or $d_n u_n > d_q u_q$

or $u_n > d_q u_q \frac{1}{d_n}$, as $d_n > 0$

or $\sum_{n=1}^{\infty} u_n > d_q u_q \sum_{n=1}^{\infty} \frac{1}{d_n}$

But $\sum \frac{1}{d_n}$ is divergent and $d_q u_q$ is a fixed number

Hence $\sum u_n$ is also divergent

DATE: / /
PAGE:

Theorem:- State and prove Pringsheim's theorem.

Statement:- If Σu_n be a convergent series of positive decreasing terms, then $\lim_{n\to\infty} \{n u_n\} = 0$

Proof:— Since Σu_n is a series of positive decreasing terms, therefore,

$$u_{n+1} \geq u_{2n}$$
$$u_{n+2} \geq u_{2n}$$
$$u_{n+3} \geq u_{2n}$$
$$\cdots\cdots$$
$$u_{2n} = u_{2n}$$

Adding $u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{2n} \geq n u_{2n}$.

Since Σu_n is convergent, by the general condition for convergence.

$(u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{2n}) \to 0$ as $n \to \infty$

$\therefore \lim_{n\to\infty} (n u_{2n}) = 0$

or $\lim_{n\to\infty} (2n u_{2n}) = 0$ ——— (1)

Hence Pringsheim's theorem is proved when n is even

Now $u_{2n+1} \leq u_{2n} \Rightarrow (2n+1) u_{2n+1} \leq (2n+1) u_{2n}$

$\Rightarrow (2n+1) u_{2n+1} \leq \frac{2n+1}{2n} \cdot (2n u_{2n})$

$\Rightarrow (2n+1) u_{2n+1} \leq \left(1 + \frac{1}{2n}\right)(2n u_{2n})$

$\Rightarrow \lim_{n\to\infty} [(2n+1) u_{2n+1}] \leq \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right) \left[\lim_{n\to\infty} (2n u_{2n})\right]$

$\Rightarrow \lim_{n\to\infty} [(2n+1) u_{2n+1}] = 0$ from (1)

Hence Pringsheim's theorem is proved when n is odd.

Hence $\lim_{n\to\infty} (n u_n) = 0$

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Theorem:- State and Prove De Morgan's and Bertrand's Test.

Statement:— The positive term series $\sum u_n$ converges or diverges according as

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] > 1 \text{ or } < 1$$

Proof:— We compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$ where $v_n = \frac{1}{n(\log n)^p}$

We know that $\sum v_n$ is convergent or divergent according as $p > 1$ or < 1

Hence, by comparison test, the given series $\sum u_n$ is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \frac{\frac{1}{n(\log n)^p}}{\frac{1}{(n+1)\{\log(n+1)\}^p}}$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left[\frac{\log\{n(1+\frac{1}{n})\}}{\log n}\right]^p$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \log(1+\frac{1}{n})}{\log n}\right\}^p$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \cdots \infty}{\log n}\right\}^p$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots \text{to } \infty\right\}^p$$

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1 + \frac{p}{n\log n} - \cdots \text{to } \infty\right)$$

By Binomial theorem

$$\text{or } \frac{u_n}{u_{n+1}} > \text{ or } < 1 + \frac{1}{n} + \frac{p}{n\log n} + \frac{p}{n^2\log n} + \cdots \text{to } \infty$$

$$\text{or } \frac{u_n}{u_{n+1}} - 1 > \text{ or } < \frac{1}{n}\left\{1 + \frac{p}{\log n} + \frac{p}{n\log n} + \cdots \text{to } \infty\right\}$$

Teacher Signature

or $x\left(\frac{4x}{4x+1}-1\right) >$ or $< 1+\frac{p}{\ln x}+\frac{p}{x\ln x}+\cdots+\infty$

or $\left\{x\left(\frac{4x}{4x+1}-1\right)-1\right\}\ln x >$ or $p\left(1+\frac{1}{x}+\cdots+\infty\right)$

Proceeding to limit as $x\to\infty$, we get—

$$\lim_{x\to\infty}\left[\left\{x\left(\frac{4x}{4x+1}-1\right)-1\right\}\log x\right] > \text{ or } < p$$

or $$\lim_{x\to\infty}\left[\left\{x\left(\frac{4x}{4x+1}-1\right)-1\right\}\log x\right] > \text{ or } < 1$$

Hence the test