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NOTES OF MATHS CORE-02 (Sem II)

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Q. Find the nth roots of unity.

Let $x = (1)^{1/n}$

So that $x^n = 1$

$\Rightarrow x^n = \cos 2r\pi + i \sin 2r\pi$

$x = [\cos 2r\pi + i \sin 2r\pi]^{1/n}$

$= \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$ — (1)

where $r = 0, 1, 2, 3 \ldots (n-1)$ in (1)

Let nth roots be $x_1, x_2, x_3 \ldots x_n$

$x_1 = \cos 0 + i \sin 0 = 1$

$x_2 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$

$x_3 = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n}$

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$x_n = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}$

If we set $w = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$

then the nth roots of unity are $1, w, w^2 \ldots w^{n-1}$

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Q. (1) Find all the roots of the eq $x^4 = 1$

We have $x^4 = 1$

$= \cos 2n\pi + i \sin 2n\pi$

$\Rightarrow x = \{\cos 2n\pi + i \sin 2n\pi\}^{1/4}$

$= \cos \frac{2n\pi}{4} + i \sin \frac{2n\pi}{4}$

Put $n = 0, 1, 2, 3$

Let the four roots be x_1, x_2, x_3, x_4

$x_1 = \cos 0 + i \sin 0$, $x_2 = \cos \frac{2\pi}{2} + i \sin \frac{2\pi}{2}$, $x_3 = \cos \frac{4\pi}{4} + i \sin \frac{4\pi}{4}$

$= 1 + i \times 0$ $= 0 + i \times 1$ $= -1 + i0 = -1$

$= 1$ $= i$,

$x_4 = \cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4}$

$= 0 - i = -i$

Thus the four roots are $1, -1, i, -i$

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Case III when n is a fraction, positive or negative

Let $n = p/q$ in its lowest terms where p is an integer either positive or negative and q is a positive Integer

Then $\left\{\cos p\theta/q + i \sin p\frac{\theta}{q}\right\}^q = \cos q \times \frac{p\theta}{q} + i \sin\left(q \cdot \frac{p\theta}{q}\right)$

$= \cos p\theta + i \sin p\theta$

Taking q th roots of both sides

$\cos p\frac{\theta}{q} + i \sin p\frac{\theta}{q}$ is one of the q th roots of

$\cos p\theta + i \sin p\theta$ (By Case I and II)

But $\cos p\theta + i \sin p\theta = (\cos\theta + i \sin\theta)^p$

Hence $\cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta$ is one of the q th roots of $(\cos\theta + i \sin\theta)^p$

$\Rightarrow \cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i \sin\theta)^{p/q}$

Therefore one of the values of $(\cos\theta + i \sin\theta)^n$ is $\cos n\theta + i \sin n\theta$

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Therefore one of the values of $(\cos\theta + i \sin\theta)^n$ is $\cos n\theta + i \sin n\theta$

The sum of nth power of roots

$$= 1 + \alpha^n + \alpha^{2n} + \alpha^{3n} + \dots + \alpha^{6n}$$

$$= \frac{1-(\alpha^n)^7}{1-\alpha^n} \quad [\text{from G.P formula}]$$

$$= \frac{1-(\alpha^7)^n}{1-\alpha^n} \longleftarrow (1)$$

Case I When n is not a multiple of 7

In this case $\alpha^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7$

$$= \cos 2\pi + i\sin 2\pi = 1$$

Numerator $= 1-(\alpha^7)^n = 1-1 = 0$

Again $\alpha^n = \left\{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right\}^n = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$

$\neq 1$

Since n is not multiple of 7

Hence the denominator $\neq 0$

Hence from (1) we have $\frac{0}{1-\alpha^n} = 0$

Case II When n is a multiple of 7

Let $n = 7m$

$$\alpha^n = \left\{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right\}^n = \cos 7m \times \frac{2\pi}{7} + i\sin 7m \times \frac{2\pi}{7}$$

$$= \cos 2m\pi + i\sin 2m\pi$$

$$= 1$$

Similarly $\alpha^{2n} = 1, \alpha^{3n} = 1$

Hence the Sum of the roots $= 7$

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Q. If $\log(x+iy) = p+iq$ then show that $y = x\tan\theta$ where $2\theta = \tan q \cdot \log(x^2+y^2)$

From the given Relation we have

$\log\left\{\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right\} = p+iq$

Let $\frac{1}{2}\log(x^2+y^2) = \alpha$ and $\tan^{-1}\frac{y}{x} = \beta$

$\therefore$ L.H.S $= \log(\alpha+i\beta)$

$= \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$ which is $p+iq$

Now equating the Imaginary parts we get

$q = \tan^{-1}\frac{\beta}{\alpha}$

$\Rightarrow \tan q = \frac{\beta}{\alpha} = \frac{\tan^{-1}\frac{y}{x}}{\frac{1}{2}\log(x^2+y^2)}$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{y}{x} = \tan q \log(x^2+y^2)$

$= \frac{1}{2}\theta$

$\therefore \tan^{-1}\frac{y}{x} = \tan\theta$

$y = x\tan\theta$

Q. Prove that $i^i = e^{-(4n+\frac{1}{2})\pi/2}$

We have $i = \cos\pi/2 + i\sin\pi/2 = e^{i\pi/2}$

$= e^{i\pi/2} \times e^{2n\pi i} = e^{i\frac{(4n+1)\pi}{2}}$

$\therefore \log i = \log e^{i(4n+1)\pi/2} = i\frac{(4n+1)\pi}{2}$

Now $i^i = e^{i\log i} = e^{i \times i\frac{(4n+1)\pi}{2}} = e^{-\frac{(4n+1)\pi}{2}}$

Proved

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Q. If $A = \{1,2,3,4\}$, define relation which have the properties of being

(i) Reflexive, Transitive but not Symmetric.

(ii) Symmetric. But neither reflexive nor Transitive

(iii) reflexive, Symmetric and Transitive.

(iv) Symmetric, Transitive But not reflexive.

Soln: Here $A \times A = \{1,2,3,4\} \times \{1,2,3,4\}$

$= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),$
$(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

(i) The Relation $R_1 = \{(1,1),(2,2),(3,3),(4,4),(2,3),(3,4),(2,4)\}$

is reflexive. because $(x,x) \in R_1 ; x = 1,2,3,4$

is transitive because $(2,3) \in R_1, (3,4) \in R_1 \Rightarrow (2,4) \in R_1$

But is not Symmetric because $(2,3) \in R_1$ But $(3,2) \in R_1$

(ii) The Relation $R_2 = \{(2,2),(3,4),(4,3),(2,3),(3,2)\}$

is Symmetric because $(3,4) \in R_2 \Rightarrow (4,3) \in R_2$

$(2,3) \in R_2 \Rightarrow (3,2) \in R_2$

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$(1,1) \notin R_2$

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(iii) The Relation $R_3 = \{(1,1),(2,2),(3,3),(4,4),(2,4),(4,2)\}$

is reflexive, Symmetric and transitive.

(iv) The relation $R_4 = \{(2,3),(3,2),(3,3),(2,2)\}$

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Theorem — The Relation 'Congruence Modulo n' in the set of integers is an Equivalence Relation.

Proof

(i) Since $a-a=n.0 \therefore a \equiv a \pmod n$ (reflexive)

(ii) If $a \equiv b \pmod n$, we have $a-b=nq$, $q \in I$
$\therefore b-a=n(-q)$ i.e $b \equiv a \pmod n$ (Symmetric)

(iii) If $a \equiv b \pmod n$ and $b \equiv c \pmod n$
then we have $a-b=nq$ and $b-c=nr$
where $q, r \in I$ therefore $a-c=n(q+r)$
where $q+r$ is also an Integer.
Hence $a \equiv c \pmod n$ (Transitive)

Thus we see that all the three properties are satisfied

Therefore, the relation 'Congruence modulo n' is an equivalence relation ✓

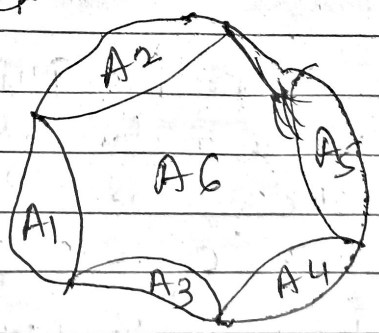
Define Partition of a set

Definition: — A partition of a set is a disjoint Class C of non-empty subset of A whose union is the full set A itself. The subset are Called the partition Sets.

Thus if A_r be a non-empty set for each value of r, then the family of non-empty set $\{A_r\}$ is Called partition of A if

(i) $\cup A_r = A$

(ii) $A_r \cap A_s = \phi$ for $r \neq s$



i.e the subsets are pairwise disjoint.

The first Condition guarantees that every element of A is one of the subsets in A. and the Second Condition guarantees that every element of A is in only one subset of A.

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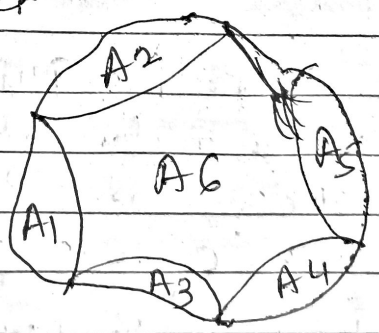
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Q: Methods to test one-one Mapping.

(1) Let $x_1, x_2 \in A$

Consider $f(x_1) = f(x_2)$

If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then $f(x)$ is a one-one function.

If $f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$ then $f(x)$ is not a one-one function.

Methods of Test onto function

Let $f: A \to B$ be a function. Let y be any arbitrary element in B satisfied $y = f(x)$ where $x \in A$. Then express x in term of y. Since x is real, the Equation which express x in terms of y will determine the range of f. If the range of f = Co-domain i.e $f(A) = B$ then the function is onto. otherwise is not onto.

Q1- Let $f: A \to B$ be defined by $f(x) = \frac{x-2}{x-3}$ where $A = R - [3]$ and $B = R - \{1\}$

Show that f is one-one and onto.

Sol we have $y = f(x) = \frac{x-2}{x-3}$

Clearly the domain of $f = R - \{3\}$

(1) To prove f is one-one

Let $x_1, x_2 \in$ domain of f. Consider

$f(x_1) = f(x_2)$

Now $f(x_1) = f(x_2)$

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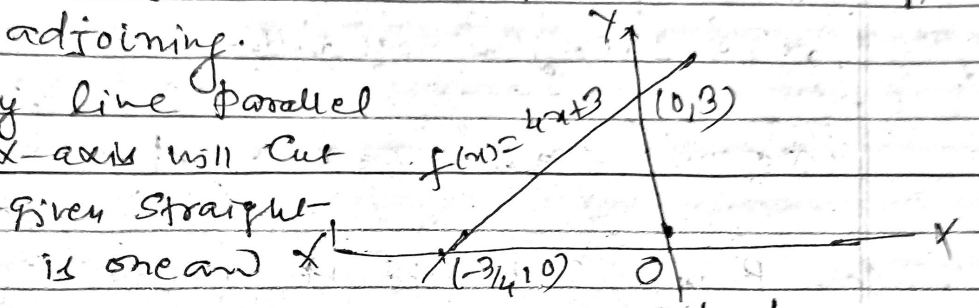
Now $f(x_1) = f(x_2)$

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Q. Let $f: R \to R$ be defined by $f(x) = 4x+3$. Show that f is invertible. Also find $f^{-1}: R \to R$.

Soln:- We shall show that f is one-one and onto so that its inverse exists.

(a) Graphically: The function $y = f(x) = 4x+3$ which represents a straight line can be given as adjoining.



Any line parallel to x-axis will cut the given straight line in one and one point only. Hence f is one-one. Also range of f is R. i.e. $(-\infty, \infty)$ which is = Co-domain of f. Hence f is onto.

(b) Analytically: Let $f(x) = f(y)$

$\Rightarrow 4x+3 = 4y+3$

$\Rightarrow x = y$

Therefore f is one-one.

Let us suppose that $b \in R$ (Co-domain)

Let its Pre-image in the domain R be a so that

$f(a) = b$

$\Rightarrow 4a+3 = b$

$\Rightarrow 4a = b-3$

$\Rightarrow a = \frac{b-3}{4} \in R$ Hence f is one-one onto.

f is invertible.

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Then $f(x) = y$

$\Rightarrow 4x+3 = y$

$\Rightarrow x = \frac{y-3}{4}$

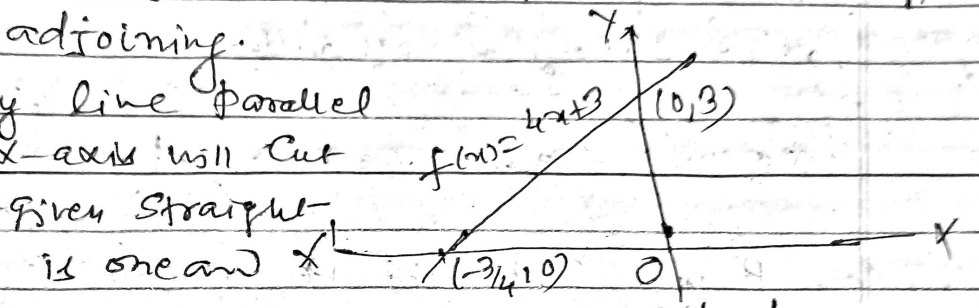
$\Rightarrow g(y) = \frac{y-3}{4}$ $\therefore f^{-1}(y) = \frac{y-3}{4}$

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Then $f(x) = y$

$\Rightarrow 4x+3 = y$

$\Rightarrow x = \frac{y-3}{4}$

$\Rightarrow g(y) = \frac{y-3}{4}$ $\therefore f^{-1}(y) = \frac{y-3}{4}$

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Matrix

Define Rank of Matrix

Definition :— Let A be a matrix of order $m \times n$ the number r is said to be the rank of Matrix A if (i) There exist at least one minor of A of order r which does not vanish
(ii) Each minor of $(r+1)$ vanishes.

Briefly we may say that the rank of a matrix is the largest order of non vanishing minor of the matrix.

Hence we can say that the rank of a matrix A is said to be r if

~~(i) The rank of every~~ non Singular ~~Square matrix A of order~~

(i) There exists one Square Submatrix of order r which is non Singular.
(ii) Every Square Submatrix of order $(r+1)$ is Singular

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ we have $|A| = 2 \neq 0$
So the rank of Matrix $A = 3$.

(ii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ Here $|A| = 0$ but there is at least one non-zero minor of order 2, Hence the Rank of Matrix $\rho(A) = 2$

(iii) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ It is 3×4 Matrix. If 3-3 rows and 4-3 Column are omitted, we get a Square Matrix whose Corresponding determinant will also be zero

Now Consider the Result of omitting (3-2) rows and (4-2) Column we obtained the determinant $\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \neq 0$

Thus $\rho(A) = 2$

Q. Find the rank of A, where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

We have $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

By $R_2 \to (-1)R_2$, $R_3 \to (-1)R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 5 \\ 0 & 6 & 4 & 22 \end{bmatrix}$$

By $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Here total number of rows = 3 (m)

k = total number of rows which contain all zero element = 0

$\therefore \rho(A) = m - k = 3 - 0 = 3$

Hence $\rho(A) = 3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & 3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad [\text{by } R_2 \to R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} \quad [\text{by } R_3 \to R_3 - 4R_2, \; R_4 \to R_4 - 9R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} \quad [\text{by } C_3 \to C_3 + 6C_2, \; C_4 \to C_4 + 3C_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad [\text{by } C_3 \to \tfrac{1}{33} C_3, \; C_4 \to \tfrac{1}{22} C_4]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{by } R_4 \to R_4 - 2R_3]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \quad [\text{by } C_4 \to C_4 - C_3]$$

which is $\begin{bmatrix} I_3 & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 1} \end{bmatrix}$

Hence the rank of matrix is 3

Teacher Signature