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Q→ Define different kinds of Elastic Constants. Analyse the relation between elastic constants. What is the limiting value of Poisson's ratio.

Ans:- Different kinds of elastic constant:-

(i) Youngs Modulus of Elasticity Y:-

Youngs modulus is defined as the ratio of longitudinal stress to longitudinal strain within the elastic limit. It is usually denoted by the letter Y.

$$\text{Youngs Modulus of Elasticity } Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

If F be the force applied normally to a cross sectional area 'a', the stress is F/a. And if, there be change l, produced in the original length L the strain is given by l/L so that

$$\text{Youngs Modulus of Elasticity} = \frac{F/a}{l/L} = \frac{F.L}{a.l}$$

(ii) Poisson's Ratio 'σ':-

The ratio of the lateral strain to longitudinal strain is constant for the material and is known as poisson's ratio. It is

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usually denoted by σ

$\therefore$ Poisson's Ratio 'σ'

$$= \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

If l and r be the initial length and radius, Δl and Δr be the increase and decrease in them respectively, we have

Longitudinal strain $= \frac{\Delta l}{l}$

Lateral strain $= \frac{\Delta r}{r}$

$\therefore$ Poisson's Ratio $= \frac{\Delta r / r}{\Delta l / l}$

$$= \frac{\Delta r}{\Delta l} \cdot \frac{l}{r}$$

Poisson's ratio have no dimension

(iii) Bulk modulus of Elasticity K :-

It is ratio of the uniform bulk stress to bulk strain. It is usually denoted by the letter K

$\therefore$ Bulk modulus of elasticity K

$$= \frac{\text{Bulk stress}}{\text{Bulk strain}}$$

Thus, if F be the force applied uniformly and normally on the surface area a, the stress is F/a and if v be the change in volume produced in original volume V the strain is $\frac{v}{V}$ and therefore

Bulk modulus of elasticity K

$$= \frac{F/a}{v/V}$$

$$= \frac{F}{a} \cdot \frac{V}{v}$$

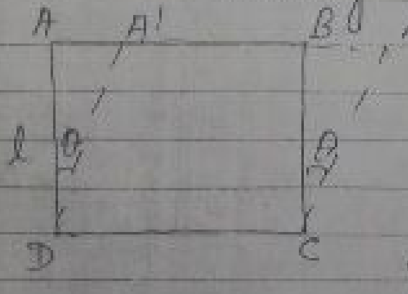
$$= \frac{PV}{v} \quad \left[\because P = \frac{F}{a} \right]$$

(iv) Modulus of rigidity η :-

The ratio of the shearing stress to the shearing strain is called the modulus of rigidity. It is usually represented by the letter η (Eta).

Modulus of rigidity η

$$= \frac{\text{Shearing stress}}{\text{Shearing strain}}$$



If a be the area of the upper surface and θ be the angle through which the vertical side AD and BC are turned, we have tangential

Shearing stress $= \frac{F}{a}$

Shearing strain $= \theta = \frac{x}{l}$

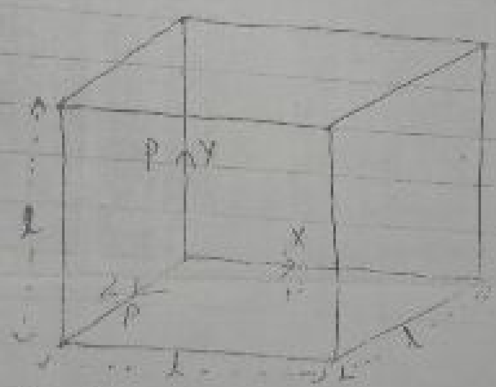
∴ Modulus of rigidity.

$$\eta = \frac{F/a}{\theta}$$

$$= \frac{F/a}{x/l}$$

$$= \frac{F}{a} \cdot \frac{l}{x}$$

Relation between different elastic constants of a substance :-



To deduce the necessary relation between the different elastic constants of a substance. We consider a unit cube of the substance which is subjected to a bulk stress P i.e the same stress (P) acts along the three axes (OX, OY, OZ) of the cube simultaneously. Since the original diamension of the substance is unity so the change in diamension will be equal to the strain produced.

From definition, Y

$$= \frac{\text{Longitudinal stress (F)}}{\text{Longitudinal strain}}$$

$\therefore$ Longitudinal strain $= + \dfrac{P}{Y}$

$\because$ Poisson's ratio 'σ'

$$= \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$\therefore$ Lateral strain $= \sigma \times$ longitudinal strain

$$= -\sigma \cdot \frac{P}{Y}$$

The elongation is taken to be positive whereas contraction is taken negative.

In order to calculate the strain produced in the cube by the application of the bulk stress (+P). Let us suppose that the stress (+P) acts along the three axis in succession and the corresponding strains produced along the three axis are represented in a tabular form below.

Stress applied along			Strain produced along		
OX	OY	OZ	OX	OY	OZ
+P	0	0	$+P/Y$	$-\sigma P/Y$	$-\sigma P/Y$
0	+P	0	$-\sigma P/Y$	$+P/Y$	$-\sigma P/Y$
0	0	+P	$-\sigma P/Y$	$-\sigma P/Y$	$+P/Y$
on adding vertically +P	+P	+P	$+\frac{P}{Y}(1-2\sigma)$	$+\frac{P}{Y}(1-2\sigma)$	$+\frac{P}{Y}(1-2\sigma)$

Thus under the action of the bulk stress (+p) each side of the cube is elongated in length by $\frac{p}{Y}(1-2\sigma)$.

$\therefore$ Length of each side of the elongated cube $= 1 + \frac{p}{Y}(1-2\sigma)$

$\therefore$ Volume of the deformed cube

$$= \left\{1 + \frac{p}{Y}(1-2\sigma)\right\}^3$$

$$= 1 + \frac{3p}{Y}(1-2\sigma) + \cdots\cdots$$

$$= 1 + \frac{3p}{Y}(1-2\sigma)$$

$\therefore$ Change in volume of the cube

$=$ Bulk strain produced

$$= 1 + \frac{3p}{Y}(1-2\sigma) - 1$$

$$= \frac{3p}{Y}(1-2\sigma)$$

$\therefore$ Bulk modulus of elasticity K

$$= \frac{\text{Bulk stress}}{\text{Bulk strain}}$$

$$= \frac{p}{\frac{3p}{Y}(1-2\sigma)}$$

$$= \frac{Y}{3(1-2\sigma)}$$

$$\therefore Y = 3K(1-2\sigma) \quad \cdots\cdots (1)$$

Let us now suppose that the cube is subjected to a shearing stress P which is equivalent to a stress +P and a stress −P acting at right angles the corresponding strain produced are shown below in the table.

OX	OY	OZ	OX	OY	OZ
+P	0	0	$+P/Y$	$-\sigma P/Y$	$-\sigma P/Y$
0	−P	0	$+\sigma \frac{P}{Y}$	$-P/Y$	$+\sigma P/Y$
On adding // vertically	−P	0	$+\frac{P}{Y}(1+\sigma)$	$-\frac{P}{Y}(1+\sigma)$	0

$\therefore$ shearing strain

$$= \text{Elongational strain} + \text{contractional strain}$$

$$= \frac{P}{Y}(1+\sigma) + \frac{P}{Y}(1+\sigma)$$

$$= \frac{2P}{Y}(1+\sigma)$$

$\therefore$ Modulus of rigidity

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}}$$

$$= \frac{P}{\frac{2P}{Y}(1+\sigma)}$$

$$= \frac{Y}{2(1+\sigma)}$$

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$\therefore$ $Y = 2\eta(1+\sigma)$ . . . . . (ii)

$\therefore$ $3K(1-2\sigma) = 2\eta(1+\sigma) = Y$ . . . (iii)

Limiting value of Poisson's Ratio:-

We have proved above that

$3K(1-2\sigma) = 2\eta(1+\sigma) = Y$

where K and η are essentially positive quantity.

In the limit $Y = 0$ [since body can not deformed $\therefore Y = 0$]

$\therefore$ $3K(1-2\sigma) = 0$ or, $2\eta(1+\sigma) = 0$

or $(1-2\sigma) = 0$ $(\because 3K \neq 0)$ or, $(1+\sigma) = 0$ $(\because \eta \neq 0)$

or, $2\sigma = 1$ or, $\sigma = -1$

or $\sigma = +\frac{1}{2}$ or, $\sigma = -1$

$\therefore$ The limiting value of σ lie between -1 to $+\frac{1}{2}$.

Q. Derive Lorentz's Transformation Equation.

Ans:→ <u>Lorentz's Transformation Equation.</u>

Suppose that S and S' are two inertial frames. S is at rest and S' is moving with a constant velocity v along the common x - axis. To simplify the problem we assume that the x-axis of both the frames is along the direction of travel of the primed fram (S'). As there is no motion along Y - axis and Z - axis, Y & Z co-ordinate will transform unchanged i.e. $y = y'$, $z = z'$. but x must transform linearly into x' and t' and t must transform linearly into x' & t'.

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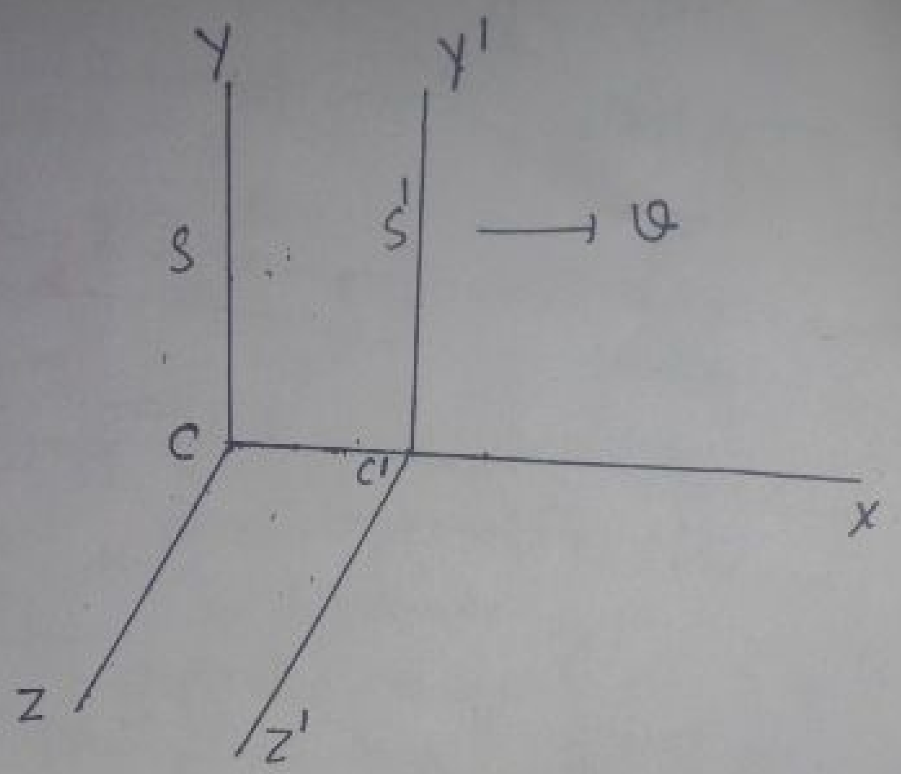
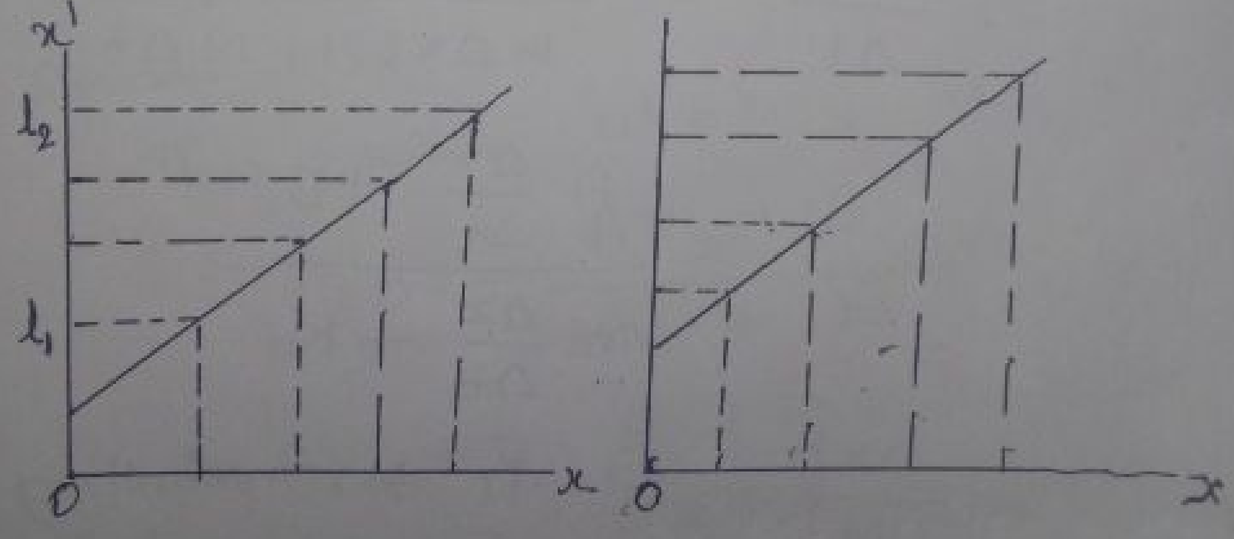


This is needed for homogeneity of space & time. When the transformation is linear, the length of a rod will not depend on what region of space it is located. We measure the length or distance by comparing with a 'measuring rod'. So it is necessary that if $l_1 = l_2$ where l_1 is the length of the rod and l_2 is the reading on the meter scale in one frame (~~the~~ unprimed frame). In the primed frame also $l_1' = l_2'$. So that the rod and scale

Contract or elongate equally and an observer get the same reading on the scale. If the transformation is nonlinear then it follows that if $l_1 = l_2$, $l_1' \neq l_2'$ i.e the length of the rod would depend upon the region of space where it is located. This is against the principle of homogeneity of space.



A similar argument can be made for time. Therefore, we seek for relativistic transformation in forms

$$x' = Ax + Bt$$

$$t' = Mx + Nt$$

where A, B, M & N are constants to be determined.

$$\therefore \Delta x' = A\Delta x + B\Delta t$$

$$\Delta t' = M\Delta x + N\Delta t$$

$$\therefore \frac{\Delta x'}{\Delta t'} = \frac{A\Delta x + B\Delta t}{M\Delta x + N\Delta t}$$

$$= \frac{A\frac{\Delta x}{\Delta t} + B}{M\frac{\Delta x}{\Delta t} + N}$$

Now $\frac{\Delta x'}{\Delta t'} = u'$, is the velocity of a primed frame.

$\frac{\Delta x}{\Delta t} = u$, is the velocity of a particle in unprimed frame.

$$\therefore u' = \frac{Au + B}{Mu + N}$$

This relation must hold good for any velocity of the particle in the primed frame and hence must hold for $u' = 0$ as well. But when $u' = 0$, the particle is at rest in the primed frame and has velocity v relative to the unprimed frame (S).

Thus when $u' = 0$, $u = v$

$$\therefore 0 = \frac{Av + B}{Mv + N}$$

$$\text{or, } Av + B = (Mv + N) \times 0$$
$$= 0$$

$$\text{or, } B = -Av$$

$$\therefore u' = \frac{A(u-v)}{Mu+N}$$

where now A, M & N are only unknowns to be determined. This relation must also hold good when the particle is at rest in the frame S. Thus $u = 0$ and $u' = -v$

$$\therefore -v = \frac{A(0-v)}{N}$$

$$\text{or } A = N$$

$$\therefore u' = \frac{A(u-v)}{Mu+N}$$

where now A & M are only two unknowns to be determined.

On the basis of the ~~principle~~ principle of constancy of velocity of light in a vaccum in all

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inertial frames, we have

$$u = u' = c$$

$$\therefore c = \frac{A(c-v)}{mc + A}$$

$$\Rightarrow m = -A\frac{v}{c^2}$$

Putting ~~th~~ value of the constant B, m and N in terms of A obtained above, we have transformation equation

$$x' = Ax - Avt$$
$$= A(x - vt)$$

$$\text{and } t' = -A\frac{v}{c^2} + At$$
$$= A\left(t - \frac{vx}{c^2}\right)$$

where A is the only one constant left that is to determined

According to first postulate of the theory of relativity i.e

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equivalence of all inertial frames, the above relations must hold good if S' is at rest and S moves with velocity $-v$.

Thus we must have

$$x = A(x' + vt')$$

and $$t = A\left(t' + \frac{vx'}{c^2}\right)$$

substituting the values of x and t, we have

$$x' = A^2\left\{x' + vt' - v\left(t' + \frac{vx'}{c^2}\right)\right\}$$

or $$x' = A^2 x'\left(1 - \frac{v^2}{c^2}\right)$$

or $$A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus finally we have transformation equations as

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \quad z = z'$$

$$t' = \frac{t - vx}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These are called Lorentz transformation equations.

Replacing primed quantities by the corresponding unprimed quantities and unprimed quantities by the corresponding primed one and v by $-v$, we obtain

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \quad z = z'$$

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$$t = \frac{t' + \frac{v x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These are called reciprocal transformation equations.

Q:- Write short notes on :- (a) gradient, (b) divergence and (c) curl.

Ans (a) Gradient :-

The gradient of a scalar point function ϕ is defined as $\nabla\phi$ and is written as grad ϕ.

$$\text{Grad}\,\phi = \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Grad ϕ is a vector quantity.

$\phi(x, y, z)$ is a function of three independent variables and its total differential $d\phi$ is

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$[\bar{r} = xi + yj + zk$
$\quad d\vec{r} = i\,dx + j\,dy + k\,dz]$

$$= \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(i\,dx + j\,dy + z\,dz)$$

$$= (\nabla\phi)\cdot dr$$

$$= |\nabla\phi|\,|dr|\cos\theta$$

The value of $d\phi$ is the greatest when $\theta = 0$. It is this property of ϕ that gives its name, the gradient of ϕ.

① **Divergence:—**

The divergence of a vector point function $\overline{F}$ is denoted by div F and is defined as below.

Let $\vec{F} = F_1 i + F_2 j + F_3 k$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (iF_1 + jF_2 + kF_3)$$

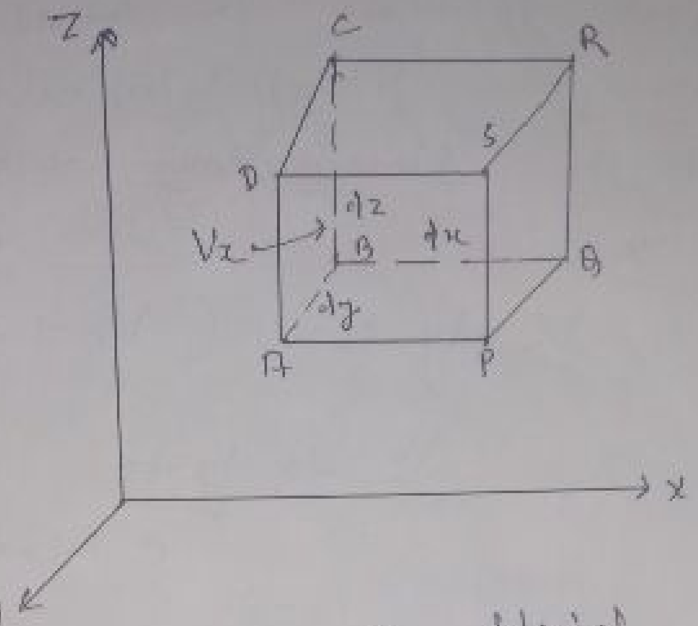
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

Physical Interpretation of Divergence:—

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to x, y and z axes respectively.

Let $\vec{V} = V_x i + V_y j + V_z k$



be the velocity of the fluid at $p(x, y, z)$.

$\therefore$ Mass of fluid flowing in through the face ABCD per time

$= \text{velocity} \times \text{Area of the face}$

$= V_x (dy\, dz)$

Mass the fluid flowing out across the face PQRS per unit time

$= V_x (x + dx)(dy\, dz)$

$= \left(V_x + \frac{\partial V_x}{\partial x} dx\right)(dy\, dz)$

Net decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time

$$= V_x \, dy \, dz - \left(V_x + \frac{\partial V_x}{\partial x} dx \right) dy \, dz$$

$$= - \frac{\partial V_x}{\partial x} dx \, dy \, dz$$

(Minus sign shows decrease)

Similarly, the decrease in mass of fluid to the flow along y-axis

$$= \frac{\partial V_y}{\partial y} dx \, dy \, dz$$

and the decrease in mass of fluid to the flow along z-axis

$$= \frac{\partial V_z}{\partial z} dx \, dy \, dz$$

Total decrease of the amount of fluid per unit time

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx \, dy \, dz$$

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (iV_x + jV_y + kV_z)$$

$$= \nabla \cdot \bar{V} = \operatorname{div} \bar{V}$$

If the fluid incompressible, there can be no gain or loss in the volume element. Hence

$$\operatorname{div} \bar{V} = 0 \quad \text{——— (1)}$$

and V is called a solenoidal vector function.

Equation (1) is also called the equation of continuity.

(c) Curl :→

The curl of a vector point function F is defined as below:

$$\operatorname{Curl} \bar{F} = \nabla \times \bar{F} \quad \text{where } \bar{F} = F_1 i + F_2 j + F_3 k$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (F_1 i + F_2 j + F_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)$$

$$+ k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

physical Meaning of curl:-

we know $V = \omega \times \bar{r}$ where ω is the angular velocity, V is the linear velocity and $\bar{r}$ is the position vector of a point on the rotating body.

$$\text{Curl } V = \nabla \times V$$

$$= \nabla \times (\omega \times \bar{r})$$

$\left[\begin{array}{l} \omega = \omega_1 i + \omega_2 j + \omega_3 k \\ r = xi + yj + zk \end{array} \right.$

$$= \nabla \times [(\omega_1 i + \omega_2 j + \omega_3 k) \times (xi + yj + zk)]$$

e.

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[(\omega_2 z - \omega_3 y) i - (\omega_1 z - \omega_3 x) j + (\omega_1 y - \omega_2 x) k \right]$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times \left[(\omega_2 z - \omega_3 y) i - (\omega_1 z - \omega_3 x) j + (\omega_1 y - \omega_2 x) k \right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_2) i - (-\omega_2 - \omega_2) j + (\omega_3 + \omega_3) k$$

$$= 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\omega$$

curl $v = 2\omega$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

If curl $\overline{F} = 0$, the field F is termed irrotational.

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